Fracture in Rock Based on the Extension Strain Failure Criterion

Sisonke Peter Lethukuthula Ndebele Ahmed Alhussein
Thabo Dimba Goitseona Theo Sono Keneilwe Raseale
Mapule Pheko Vukani Mhlongo

Supervisor: Prof. David Mason

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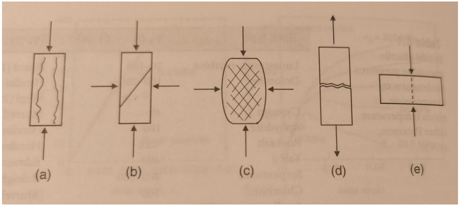
Outline

Introduction



a. Longitudinal spliting under uniaxial tension
b. shear fracture
c. multiple shear fractures
d. extension fracture
e. extension fracture produces by opposing line loads

Problem Statement

Problem: Traditional rock failure criteria (e.g., Mohr-Coulomb) are insufficient under **low-stress conditions**, especially near excavation surfaces.

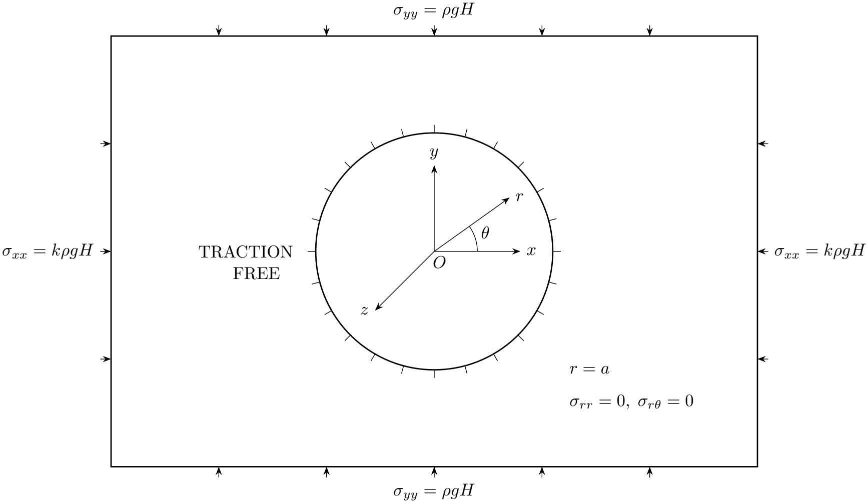
Proposed Approach:

- **Extension Strain Criterion:** Fractures initiate when *extension strain* exceeds a critical rock-specific value.
- Fractures form **normal to maximum extension strain** (least compressive principal stress).
- Includes the effect of the **intermediate principal stress**, unlike Mohr-Coulomb.

SA Mines

Rank	Name of mine	Depth	Location	Primary resource	Active / closed
1	Mponeng Gold Mine	4.0 km (2.5 mi)	South Africa	Gold[1]	Active
2	TauTona Mine	3.9 km (2.4 mi)	South Africa	Gold[1]	Active. Grouped under Mponeng [2]
3	Savuka Gold Mine	3.7 km (2.3 mi)	South Africa	Gold[1]	Closed 2017[3]
4	East Rand Mine	3.585 km (2.228 mi)[4]	South Africa	Gold	Closed 2008[2]
5	Driefontein Mine	3.420 km (2.125 mi)[5]	South Africa	Gold[1]	Active
6	Kusasalethu mine	3.388 km (2.105 mi)[6]	South Africa	Gold[1]	Active
7	Empire Mine	3.355 km (2.085 mi)	United States	Gold[7]	Closed 1956
8	Kloof mine	3.347 km (2.080 mi)[8]	South Africa	Gold, uranium	Active
9	Laronde Mine	3.260 km (2.026 mi)[9]	Canada	Gold, copper, silver, zinc[1]	Active
10	Kolar Gold Fields	3.217 km (1.999 mi)	India	Gold[1]	Closed 2001
11	Blyvooruitzicht mine	3.213 km (1.996 mi)[10]	South Africa	Gold, uranium	Active

Problem Geometry



Governing Equations

$$\nabla^4 U = 0 \tag{1}$$

Boundary conditions:

- Tunnel wall ($r = a$):

$$\sigma_{rr} = 0, \quad \sigma_{r\theta} = 0 \tag{2}$$

- Far field ($r \to \infty$):

$$U \to \frac{1}{4}(1+k)\rho g H r^2 + \frac{1}{4}(1-k)\rho g H r^2 \cos 2\theta \tag{3}$$

Solution Form

Assume:

$$U(r,\theta) = f_0(r) + f_2(r)\cos 2\theta, \tag{4}$$

where f_0 and f_2 satisfy the Euler equations,

$$r^4 \frac{d^4 f_0}{dr^4} + 2r^3 \frac{d^3 f_0}{dr^3} - r^2 \frac{d^2 f_0}{dr^2} + r \frac{df_0}{dr} = 0 \tag{5}$$

$$r^4 \frac{d^4 f_2}{dr^4} + 2r^3 \frac{d^3 f_2}{dr^3} - 9r^2 \frac{d^2 f_2}{dr^2} + 9r \frac{df_2}{dr} = 0 \tag{6}$$

Solution: Airy stress function

$$U(r,\theta) = \frac{1}{4}(1+k)\rho g H r^2 - \frac{1}{2}a^2(1+k)\rho g H \ln r$$
$$+ \left[\frac{1}{4}(1-k)\rho g H r^2 - \frac{1}{2}a^2(1-k)\rho g H + \frac{1}{4}\frac{a^4}{r^2}(1-k)\rho g H\right]\cos 2\theta \tag{7}$$

Stress Components

$$\sigma_{\theta\theta} = \frac{1}{2}\rho g H\left[(1+k)\left(1+\frac{a^2}{r^2}\right) + (1-k)\left(1+\frac{3a^4}{r^4}\right)\cos(2\theta)\right]$$

$$\sigma_{rr} = \frac{1}{2}\rho g H\left(1-\frac{a^2}{r^2}\right)\left[(1+k) - (1-k)\left(1-\frac{3a^2}{r^2}\right)\cos(2\theta)\right] \tag{8}$$

$$\sigma_{r\theta} = \frac{1}{2}\rho g H(1-k)\left(1-\frac{a^2}{r^2}\right)\left(1+\frac{3a^2}{r^2}\right)\sin(2\theta)$$

Hooke's Law

Hooke's law in Rock Mechanics notation,

$$\epsilon_{ik} = \frac{\nu}{E}(\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz}) - \left(\frac{1+\nu}{E}\right)\sigma_{ik} \tag{9}$$

Since $\epsilon_{zz} = \frac{\partial U_z}{\partial z} = 0$,

$$\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta}) \tag{10}$$

Thus,

$$\begin{aligned} \epsilon_{rr} = \frac{1}{2}\rho g H \frac{(1+\nu)}{E} \Bigg[& -(1+k)\left(1 - 2\nu - \frac{a^2}{r^2}\right) \\ & + (1-k)\left(1 - 4(1-\nu)\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right)\cos(2\theta)\Bigg] \end{aligned} \tag{11}$$

Sidewalls at $\theta = 0, \pi$

For the side walls $\theta = 0, \pi$, thus,

$$\epsilon_{rr} = \frac{1}{2}\rho g H \frac{(1+\nu)}{E} F(\bar{r}) \tag{12}$$

where,

$$F(\bar{r}) = 1-k-(1+k)(1-2\nu)+[1+k-4(1-k)(1-\nu)]\frac{1}{\bar{r}^2}+3(1-k)\frac{1}{\bar{r}^4} \tag{13}$$

and

$$\bar{r} = \frac{r}{a} \tag{14}$$

General Results

- Example (Witwatersrand area):

$$\nu = \frac{1}{4} \tag{15}$$

- For $\bar{r} = 1$:

$$F(1) = \frac{1}{2}(3 - k) \tag{16}$$

- As $\bar{r} \to \infty$:

$$F(\infty) = \frac{1}{2}(\frac{1}{3} - k) \tag{17}$$

Analysis of the Solution and Results

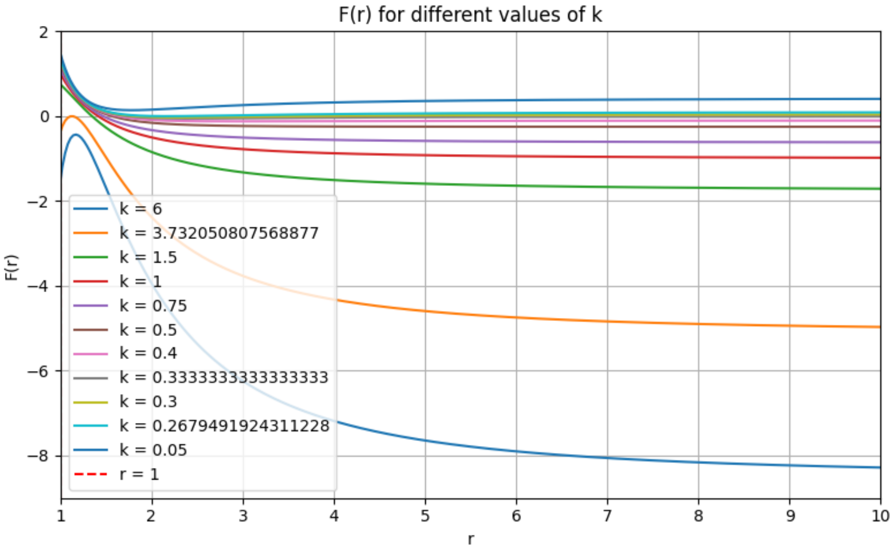
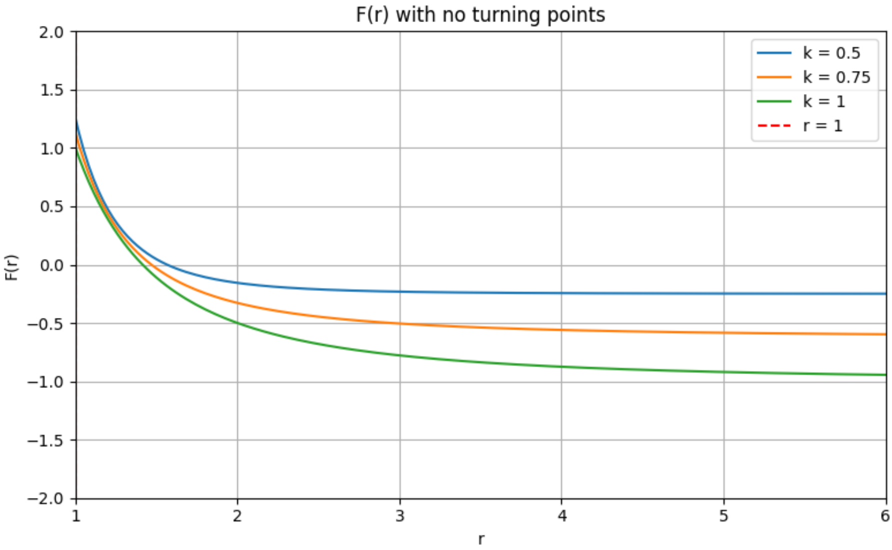


Figure: F(r) for different values of k



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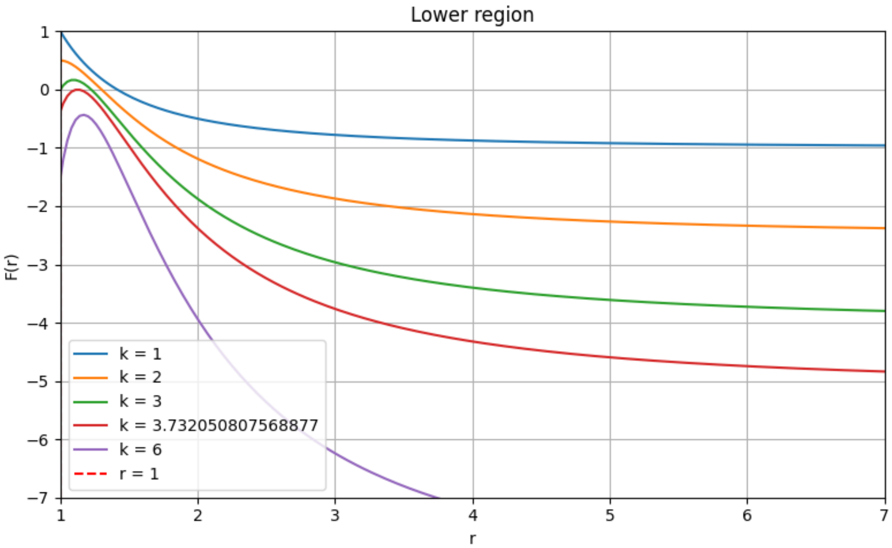

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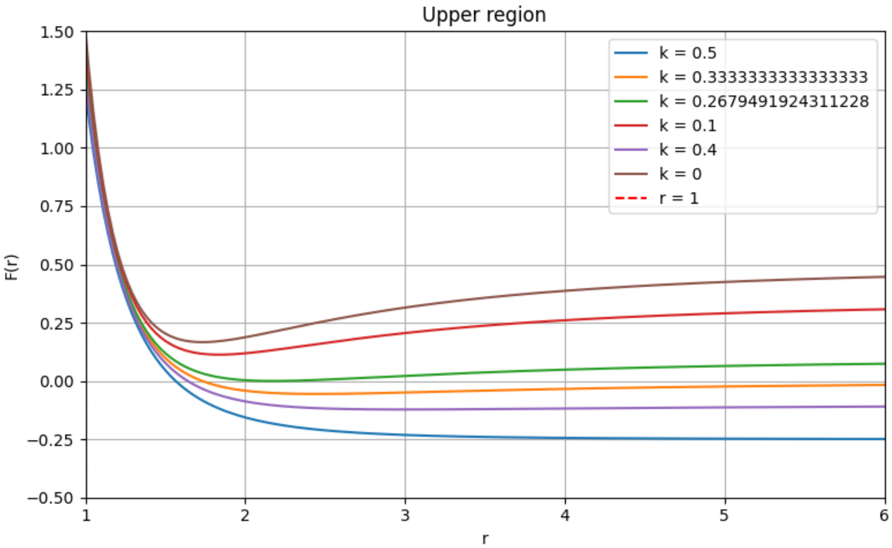

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Hanging wall at $\theta = \frac{\pi}{2}$

For the hanging wall, $\theta = \frac{\pi}{2}$, thus,

$$\epsilon_{rr}(\bar{r}, \frac{\pi}{2}) = \frac{1}{2}\rho g H \frac{(1+\nu)}{E} G(\bar{r}) \tag{18}$$

where,

$$G(\bar{r}) = \frac{1}{2}(k-3) + 2(2-k)\frac{1}{\bar{r}^2} - 3(1-k)\frac{1}{\bar{r}^4} \tag{19}$$

and

$$\bar{r} = \frac{r}{a} \tag{20}$$

$$\frac{d}{d\bar{r}}G(\bar{r}) = 4(k-2)\frac{1}{\bar{r}^3} + 12(1-k)\frac{1}{\bar{r}^5} \tag{21}$$

$$G(\infty) = \frac{1}{2}(k-3) \tag{22}$$

$$G(1) = \frac{3}{2}(k-\frac{1}{3}) \tag{23}$$

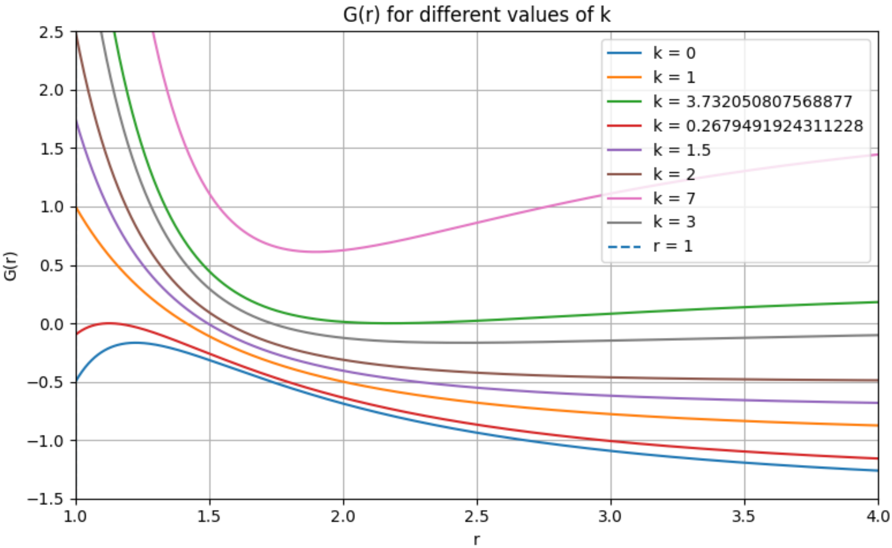
Turning points:
For $0 < k < 1$:

$$\bar{r}^2 = \frac{3(1-k)}{(2-k)} \tag{24}$$

For $1 < k < 2$: No turning points.
For $k > 2$:

$$\bar{r}^2 = \frac{3(k-1)}{(k-2)} \tag{25}$$

Analysis of the Solution and Results



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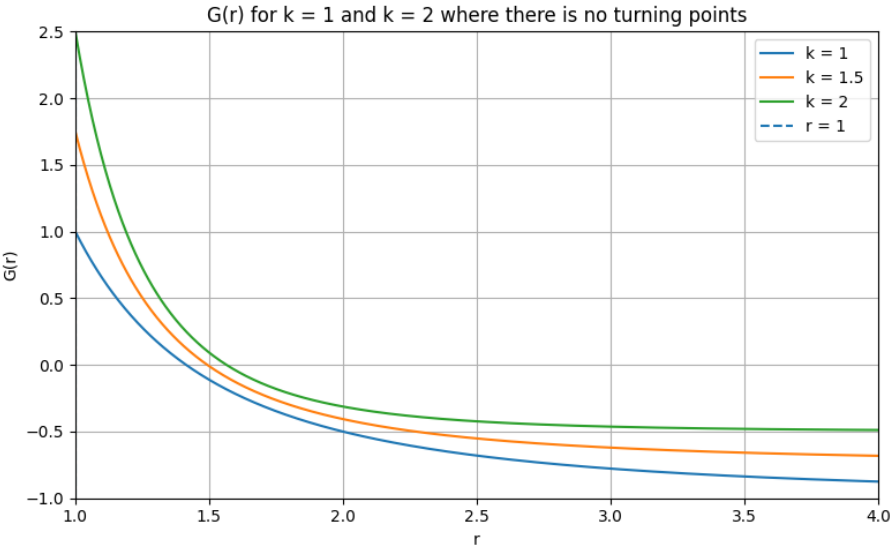

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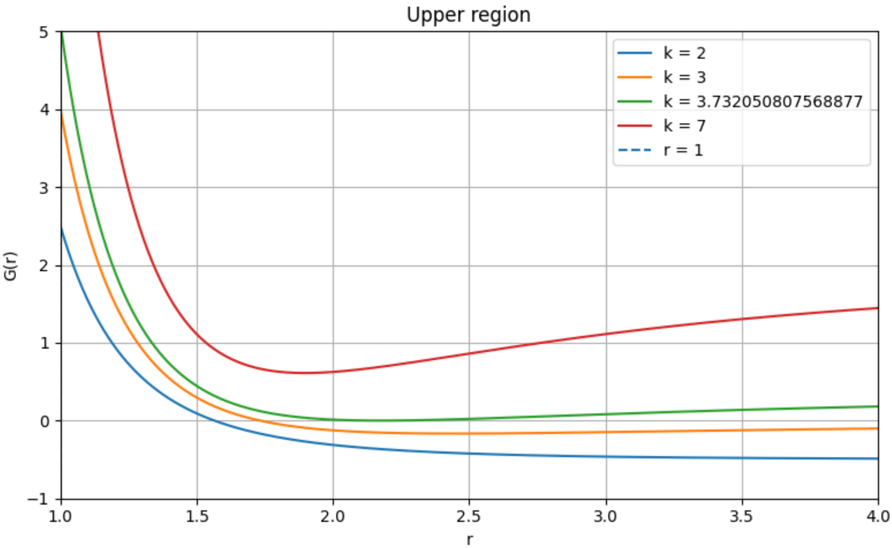

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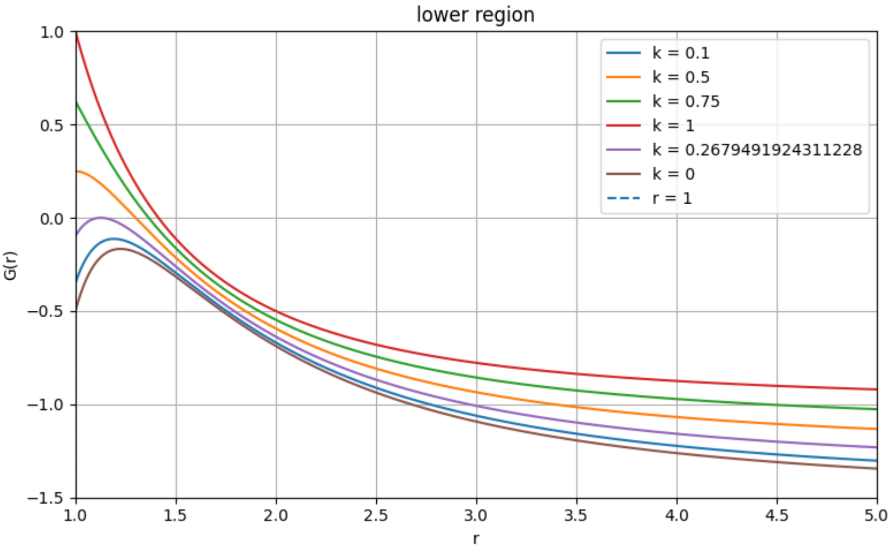

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Conclusion

Sidewall $\theta = 0$:

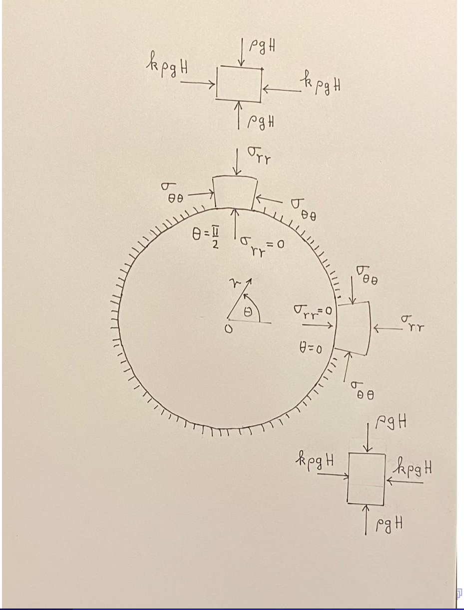
- In general extension near excavation only
- $0 \leq k \leq 2 - \sqrt{3}$ extension for $1 < \bar{r} < \infty$
- $2 + \sqrt{3} \leq k < \infty$ no extension

Hanging wall $\theta = \frac{\pi}{2}$:

- In general, extension near excavation only
- For $k > 2 + \sqrt{3}$ extension for $1 < \bar{r} < \infty$
- $0 \leq k \leq 2 - \sqrt{3}$ no extension

Results support the extension strain failure criterion.

Cont..



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Thank You!

Questions?